1. Normally, we call $\mathbf{x}$ an eigenvector of $\mathbf{A}$ if $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, i.e. the "input" and "output" are parallel when we multiply $\mathbf{A}$ on the right by a (column) vector.

However, it is also possible to define *left-hand eigenvectors*, i.e.the (column) vector $\mathbf{y}$ is a left-hand eigenvector of $\mathbf{A}$ if, for some scalar η,

$$\mathbf{y}^T \mathbf{A} = \eta\, \mathbf{y}^T$$

Show that:

(i) $\mathbf{y}$ is a left-hand eigenvector of $\mathbf{A}$ if and only if $\mathbf{y}$ is also a "normal" (i.e. right-hand) eigenvector of $\mathbf{A}^T$.

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solution:

$$\mathbf{y}^T \mathbf{A} = \eta\, \mathbf{y}^T \quad \Longleftrightarrow \quad \left(\mathbf{y}^T \mathbf{A}\right)^T = \left(\eta\, \mathbf{y}^T\right)^T \quad \Longleftrightarrow \quad \mathbf{A}^T \mathbf{y} = \eta\, \mathbf{y}$$

But, by definition, the last condition means precisely that $\mathbf{y}$ is an eigenvector of $\mathbf{A}^T$, with η as the associated eigenvalue.

(ii) The eigenvalues of $\mathbf{A}$ and $\mathbf{A}^T$ are identical. (Hint: Use properties of the determinant on the respective characteristic polynomials.)

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solution:

$$\mathbf{A}^T \mathbf{y} = \eta\, \mathbf{y} \quad \Longleftrightarrow \quad \det\left(\mathbf{A}^T - \eta\mathbf{I}\right) = 0$$

But we know, from the properties of determinants, that $\det(\mathbf{B}) = \det\left(\mathbf{B}^T\right)$. Therefore:

$$\det\left(\mathbf{A}^T - \eta\mathbf{I}\right) = \det\left(\left(\mathbf{A}^T - \eta\mathbf{I}\right)^T\right) = \det\left(\left(\mathbf{A}^T\right)^T - \eta\left(\mathbf{I}\right)^T\right)$$
$$\equiv \det\left(\mathbf{A} - \eta\mathbf{I}\right)$$

Hence both $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic polynomial. But, since the roots of a polynomial are unique, this means $\mathbf{A}$ and $\mathbf{A}^T$ must have the same eigenvalues.

(iii) What, if any, relationship exists between the eigenvectors of $\mathbf{A}$ and those of $\mathbf{A}^T$ when all of the eigenvalues are *distinct.*

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solution:

We know that if all the eigenvalues of $\mathbf{A}$ are distinct, then $\mathbf{A}$ will have a full set of linearly independent eigenvectors and be diagonalizable, i.e. there will be a matrix $\mathbf{V}$ such that
$$\mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V} = \mathbf{D}$$

where $\mathbf{D}$ is a diagonal matrix, and the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}$. But this implies

$$\left(\mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}\right)^T = \mathbf{D}^T \quad \Longleftrightarrow \quad \mathbf{V}^T\,\mathbf{A}^T\,\left(\mathbf{V}^{-1}\right)^T = \mathbf{D}$$

(since $\mathbf{D}$ is diagonal, it's trivially symmetric.) But

$$\left(\mathbf{V}^{-1}\right)^T = \left(\mathbf{V}^T\right)^{-1}$$

(simply transpose $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$). Therefore:

$$\mathbf{V}^T\,\mathbf{A}^T\,\left(\mathbf{V}^{-1}\right)^T = \mathbf{D} \quad \Longleftrightarrow \quad \mathbf{A}^T\,\left(\mathbf{V}^{-1}\right)^T = \mathbf{D}\left(\mathbf{V}^T\right)^{-1} = \mathbf{D}\left(\mathbf{V}^{-1}\right)^T$$

Which immediately implies that the columns of $\left(\mathbf{V}^{-1}\right)^T$ are the eigenvectors of $\mathbf{A}^T$. Therefore, $\mathbf{y}$ is an eigenvector of $\mathbf{A}^T$, if and only if $\mathbf{y}^T$ is a row of $\mathbf{V}^{-1}$, where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{A}$.

But $\mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$, i.e. the rows of $\mathbf{V}^{-1}$ are orthogonal to every column of $\mathbf{V}$ *except* the one with the same index. (Specifically, the first row of $\mathbf{V}{-1}$ is orthogonal to every column of $\mathbf{V}$ but the first, the second row of $\mathbf{V}{-1}$ is orthogonal to every column of $\mathbf{V}$ but the second, etc.) Therefore, $\mathbf{y}^{(i)}$ is an eigenvector of $\mathbf{A}^T$ if and only if $\mathbf{y}^{(i)}$ lies in the orthogonal complement of $\mathbf{v}^{(i)}$, where $\mathbf{v}^{(i)}$ is the eigenvector of $\mathbf{A}$ associated with same eigenvalue. So, if needed, $\mathbf{y}^{(i)}$ can be found directly from $\mathbf{v}^{(i)}$ by several different methods, e.g. least squares, Gram-Schmidt, or $\mathbf{Q}\,\mathbf{R}$ factorization of $\mathbf{V}$ with the column corresponding to $\mathbf{v}^{(i)}$ removed.

2. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 3 & -1 \\ 2 & -5 & -1 \end{bmatrix}$$

Conduct five iterations of the basic power method, starting with $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, with normalization (using the infinity norm) after each step, and estimate the dominant eigenvalue and is associated eigenvector. Compare your answer with the exact (MATLAB) answer.

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solution:

By definition, the basic power method satisfies

$\quad$ (i) $\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)}$

$\quad$ (ii) $\mathbf{x}^{(k+1)} = \dfrac{\mathbf{x}^{(k+1)}}{\|\mathbf{x}^{(k+1)}\|_\infty}$

So, starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

proceed

$$\mathbf{x}^{(1)} = \mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 3 & -1 \\ 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \\ -4 \end{bmatrix}$$

Normalize

$$\mathbf{x}^{(1)} = \frac{1}{8} \begin{bmatrix} -8 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Repeat

$$\mathbf{x}^{(2)} = \mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 3 & -1 \\ 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1.5 \\ 2.5 \\ -1.5 \end{bmatrix}$$

and normalize

$$\mathbf{x}^{(2)} = \frac{1}{2.5} \begin{bmatrix} -1.5 \\ 2.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} -0.6 \\ 1.0 \\ -0.6 \end{bmatrix}$$

solution:

Continuing yields:

$$\mathbf{x}^{(3)} = \begin{bmatrix} -6.4 \\ 4.8 \\ -5.6 \end{bmatrix} \quad , \text{ normalize to } \quad \mathbf{x}^{(3)} = \begin{bmatrix} -1.000 \\ 0.750 \\ -0.875 \end{bmatrix}$$

and

$$\mathbf{x}^{(4)} = \begin{bmatrix} -4.875 \\ 5.125 \\ -4.875 \end{bmatrix} \quad , \text{ normalize to } \quad \mathbf{x}^{(4)} = \begin{bmatrix} -0.9512 \\ 1.0000 \\ -0.9512 \end{bmatrix}$$

(to four decimal places) and finally

$$\mathbf{x}^{(5)} = \begin{bmatrix} -6.0488 \\ 5.8537 \\ -5.9512 \end{bmatrix} \quad , \text{ normalize to } \quad \mathbf{x}^{(5)} = \begin{bmatrix} -1.0000 \\ 0.9677 \\ -0.9839 \end{bmatrix}$$

Observe the values of the $\mathbf{x}^{(i)}$ appear to be "settling down" (to and eigenvector). At this point the approximate eigenvector is the normalized value of $\mathbf{x}^{(5)}$, and therfore

$$\mathbf{Av} = \begin{bmatrix} 4 & -7 & -5 \\ -2 & 3 & -1 \\ 2 & -5 & -1 \end{bmatrix} \begin{bmatrix} -1.0000 \\ 0.9677 \\ -0.9839 \end{bmatrix} = \begin{bmatrix} -5.8548 \\ 5.8871 \\ -5.8548 \end{bmatrix} \doteq \lambda\mathbf{v} \doteq 5.9605 \begin{bmatrix} -1.0000 \\ 0.9677 \\ -0.9839 \end{bmatrix}$$

where we obtained the value 5.9605 as the least squares solution of

$$\mathbf{v}\lambda = \mathbf{Av} \quad \Longrightarrow \quad \lambda = \frac{\mathbf{v}^T\mathbf{Av}}{\mathbf{v}^T\mathbf{v}}$$

and so

$$\lambda_1 \doteq 5.9605 \quad , \quad \text{and} \quad \mathbf{v}^{(1)} \doteq \begin{bmatrix} -1.0000 \\ 0.9677 \\ -0.9839 \end{bmatrix}$$

Note the exact (MATLAB) values are: $\quad \lambda_1 = 6$, and $\mathbf{v}^{(1)} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

(The other two eigenvalues are $\lambda_2 = 2$ and $\lambda_3 = -2$.

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 21 & -39 \\ 6 & 28 & -42 \\ 3 & 15 & -23 \end{bmatrix}$$

Conduct five iterations of the basic power method, starting with $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, with normalization (using the infinity norm) after each step, and estimate the dominant eigenvalue and is associated eigenvector. Compare your answer with the exact (MATLAB) answer.

solution:

By definition, the basic power method satisfies

(i) $\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)}$

(ii) $\mathbf{x}^{(k+1)} = \dfrac{\mathbf{x}^{(k+1)}}{\| \mathbf{x}^{(k+1)} \|_\infty}$

So, starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

proceed

$$\mathbf{x}^{(1)} = \mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} 7 & 21 & -39 \\ 6 & 28 & -42 \\ 3 & 15 & -23 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ -8 \\ -5 \end{bmatrix}$$

Normalize

$$\mathbf{x}^{(1)} = \frac{1}{11} \begin{bmatrix} -11 \\ -8 \\ -5 \end{bmatrix} = \begin{bmatrix} -1.0000 \\ -0.7273 \\ -0.4545 \end{bmatrix}$$

(to four displayed MATLAB decimal places.) Repeat

$$\mathbf{x}^{(2)} = \mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} 7 & 21 & -39 \\ 6 & 28 & -42 \\ 3 & 15 & -23 \end{bmatrix} \begin{bmatrix} -1.0000 \\ -0.7273 \\ -0.4545 \end{bmatrix} = \begin{bmatrix} -4.5455 \\ -7.2727 \\ -3.4545 \end{bmatrix}$$

and normalize

$$\mathbf{x}^{(2)} = \frac{1}{7.2727} \begin{bmatrix} -4.5455 \\ -7.2727 \\ -3.4545 \end{bmatrix} = \begin{bmatrix} -0.6250 \\ -1.0000 \\ -0.4750 \end{bmatrix}$$

solution:

Continuing yields:

$$\mathbf{x}^{(3)} = \begin{bmatrix} -6.8500 \\ -11.8000 \\ -5.9500 \end{bmatrix} \quad , \text{ normalize to} \quad \mathbf{x}^{(3)} = \begin{bmatrix} -0.5805 \\ -1.0000 \\ -0.5042 \end{bmatrix}$$

and

$$\mathbf{x}^{(4)} = \begin{bmatrix} -5.3983 \\ -10.3051 \\ -5.1441 \end{bmatrix} \quad , \text{ normalize to} \quad \mathbf{x}^{(4)} = \begin{bmatrix} -0.5238 \\ -1.0000 \\ -0.4992 \end{bmatrix}$$

(to four decimal places) and finally

$$\mathbf{x}^{(5)} = \begin{bmatrix} -5.1990 \\ -10.1776 \\ -5.0905 \end{bmatrix} \quad , \text{ normalize to} \quad \mathbf{x}^{(5)} = \begin{bmatrix} -0.5108 \\ -1.0000 \\ -0.5002 \end{bmatrix}$$

Observe the values of the $\mathbf{x}^{(i)}$ appear to be "settling down" (to and eigenvector). At this point the approximate eigenvector is the normalized value of $\mathbf{x}^{(5)}$, and therfore

$$\mathbf{Av} = \begin{bmatrix} 7 & 21 & -39 \\ 6 & 28 & -42 \\ 3 & 15 & -23 \end{bmatrix} \begin{bmatrix} -0.5108 \\ -1.0000 \\ -0.5002 \end{bmatrix} = \begin{bmatrix} -5.0695 \\ -10.0582 \\ -5.0288 \end{bmatrix}$$

$$\doteq \lambda\mathbf{v} \doteq 10.0344 \begin{bmatrix} -0.5108 \\ -1.0000 \\ -0.5002 \end{bmatrix}$$

where we obtained the value 10.0344 as the least squares solution of

$$\mathbf{v}\lambda = \mathbf{Av} \quad \implies \quad \lambda = \frac{\mathbf{v}^T \mathbf{Av}}{\mathbf{v}^T \mathbf{v}}$$

and so

$$\lambda_1 \doteq 10.0344 \quad , \quad \text{and} \quad \mathbf{v}^{(1)} \doteq \begin{bmatrix} -0.5108 \\ -1.0000 \\ -0.5002 \end{bmatrix}$$

solution:

Note the exact (MATLAB) values are: $\lambda_1 = 10$,   and $\mathbf{v}^{(1)} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$.

(The other two eigenvalues are $\lambda_2 = 4$ and $\lambda_3 = -2$.

4. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 10 & -7 & -12 \\ 2 & 1 & -12 \\ 11 & -11 & -3 \end{bmatrix}$$

Conduct three iterations of the basic power method, starting with $\mathbf{x}^{(0)} = [\, 1 \ 1 \ 1 \,]^T$, with normalization (using the infinity norm) after each step, and show that, in this instance, the method apparently will never converge. Using the exact eigenvalues and associated eigenvectors of $\mathbf{A}$ as computed by MATLAB, explain what went "wrong" here.

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solution:

By definition, the basic power method satisfies

(i) $\mathbf{x}^{(k+1)} = \mathbf{A}\mathbf{x}^{(k)}$

(ii) $\mathbf{x}^{(k+1)} = \dfrac{\mathbf{x}^{(k+1)}}{\|\, \mathbf{x}^{(k+1)} \,\|_\infty}$

So, starting with

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

proceed

$$\mathbf{x}^{(1)} = \mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} 10 & -7 & -12 \\ 2 & 1 & -12 \\ 11 & -11 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -9 \\ -3 \end{bmatrix}$$

Normalize

$$\mathbf{x}^{(1)} = \frac{1}{9} \begin{bmatrix} -9 \\ -9 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -\frac{1}{3} \end{bmatrix}$$

(to four displayed MATLAB decimal places.) Repeat

$$\mathbf{x}^{(2)} = \mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} 10 & -7 & -12 \\ 2 & 1 & -12 \\ 11 & -11 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and observe that normalization isn't required here.

solution:

Continuing yields:

$$\mathbf{x}^{(3)} = \begin{bmatrix} -9 \\ -9 \\ -3 \end{bmatrix} \quad , \text{ normalize to } \quad \mathbf{x}^{(3)} = \begin{bmatrix} -1 \\ -1 \\ -\frac{1}{3} \end{bmatrix}$$

and by now it should be obvious that all future iterates will simply alternate back and forth between

$$\begin{bmatrix} -1 \\ -1 \\ -\frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In other words, in this case, the power method iterates will never converge. To understand why, observe that, according to MATLAB, the exact eigenvalues and associated eigenvectors are: Note the exact (MATLAB) values are:

$$\lambda_1 = 8 \quad , \quad \lambda_2 = 3 \quad \text{and} \quad \lambda_3 = -3$$

and

$$\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad , \quad \mathbf{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \text{and} \quad \mathbf{v}^{(3)} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

(Note we've "cleaned" up the eigenvectors provided by MATLAB a bit!) In this case, if we expand $\mathbf{x}^{(0)}$ in terms of the eigenvectors, we find:

$$\mathbf{x}^{(0)} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)} + \alpha_3 \mathbf{v}^{(3)}$$

$$\implies \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\implies \quad \alpha_1 = 0 \quad , \quad \alpha_2 = -1 \quad \text{and} \quad \alpha_3 = 1$$

In other words, $\mathbf{x}^{(0)}$ started out with **no component** in the direction of $\mathbf{v}^{(1)}$. Therefore, we expect the iteration to produce

$$\mathbf{x}^{(k)} = \mathbf{A}^k \mathbf{x}^{(0)} = \alpha_2 \lambda_2^k \mathbf{v}^{(2)} + \alpha_3 \lambda_3^k \mathbf{v}^{(3)} \equiv \alpha_2 (3)^k \mathbf{v}^{(2)} + \alpha_3 (-3)^k \mathbf{v}^{(3)}$$

solution:

or equivalently

$$\mathbf{x}^{(k)} = (3)^k \left\{ \alpha_2 \mathbf{v}^{(2)} + \alpha_3 (-1)^k \mathbf{v}^{(3)} \right\}$$

In other words, $\mathbf{x}^{(k)}$ will continually oscillate between the direction of

$$\alpha_2 \mathbf{v}^{(2)} + \alpha_3 \mathbf{v}^{(3)} \quad \text{and} \quad \alpha_2 \mathbf{v}^{(2)} - \alpha_3 \mathbf{v}^{(3)}$$

(In a "real" application, the effects of finite-precision errors would likely eventually introduce a "small" component in the direction of $\mathbf{V}^{(1)}$, and this would be sufficient, eventually, to cause convergence to the correct eigenvector. In fact, if we start the MATLAB calculations here with

$$\mathbf{x}^{(0)} = \mathbf{zeros}(3,1) + 100 \ eps \ \mathbf{rand}(3,1)$$

then we obtain convergence in forty to fifty iterations.)